Electromagnetic Theory of the Loosely Braided Coaxial Cable: Part I

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Abstract—A solution to Maxwell's equations subject to boundary conditions on counterwound helical wires is achieved. The helices are contained in a cylindrical surface that is concentric to a perfectly conducting center conductor of circular cross section. The permittivity of the annular region may be different from that of the external region. The excitation is taken to be symmetrical about the cable which leads to a considerable simplification of the formulation. The key step is to recognize that the assumed form of the current on the thin helical wires is a spatial harmonic expansion that leads to a doubly infinite expansion, in such harmonics, for the resultant fields. The inherent complication of the problem results from the intercoupling between the spatial harmonics of the helix currents. Various generalizations of the theory are also indicated.

Introduction

A BRAIDED coaxial cable can be envisaged as a composite counterwound helical structure with a concentric center conductor. While the actual geometry varies greatly from one cable to another, the basic concept is that each helix carries a current that interacts with neighboring helices and with the center conductor and the insulating dielectrics. Much progress has been made in understanding the operation of braided coaxial cables by postulating equivalent circuit or transmission-line parameters that characterize, in some sense, the mean electrical properties [1], [2]. An example of this approach is to represent the braided-wire sheath by a thin uniform cylindrical shell with a specified transfer impedance that relates the axial electric field and the discontinuity of the tangential magnetic field [3], [4]. Obviously, such a parameter has great utility when the performance of the cable in a complicated environment is to be determined. While the surface-transfer impedance of the sheath and related parameters can be measured, it seems that a basic electromagnetic analysis of some idealized cases is badly needed. It is really surprising that such a general analysis has not been attempted before now although some related theoretical work in connection with traveling-wave tubes has been performed [5]. Also, we should call attention to some important studies by Latham [6] and also by Lee and Baum [7] who put the transmission-line theory on a firmer basis.

Our immediate purpose, then, is to formulate the problem of a cylindrical structure that consists basically of a dielectric-coated conductor that is sheathed by a finite number of counterwound helices. Our first task will be to obtain the fields of a single helix that carries a filamental current that can be represented by a spatial harmonic expansion. We then add the fields of the counterwound helix and the prescribed incident field. An impedance boundary condition at the surface of the helical wires is then applied. The resulting infinite set of equations can be solved, in principle, for the amplitudes of the individual spatial harmonics of the filamental currents. In concept, this aspect of the problem is the same as used for determining the currents induced on a rectangular wire mesh by an incident plane wave [8], [9]. Also, it should be mentioned that Casey [10] has solved a similar problem as posed here, but he assumed initially that the filamental currents were uniform. The validity of this assumption could be questioned in the general case of counterwound helices.

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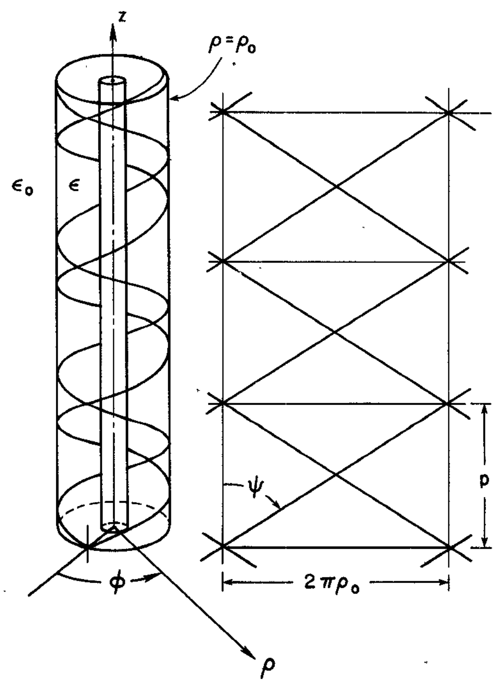

Fig. 1. Perspective view of counterwound helices and planar development of the cylindrical surface.

Basic Formulation of Problem

With respect to a right-handed cylindrical coordinate system (ρ,ϕ,z), we can define a single thin-wire helix by the equation $\phi = (z/\rho_0) \tan \psi$. Here ρ_0 is the radius of the cylindrical surface that is common to the helix and ψ is the pitch angle as illustrated in Fig. 1. The center conductor of radius a is assumed to be perfectly conducting. As indicated below, the helix wires may be imperfectly conducting and characterized by an appropriate impedance parameter that relates the filamental current to the tangential electric field. The region external to the helix (i.e., $\rho > \rho_0$) is taken to be free space with permittivity ε_0. An insulating dielectric of permittivity ε is assumed to occupy the concentric region $\rho_0 > \rho > a$. Thus we neglect any external dielectric jacket and possibly lossy external coatings although they would not introduce any new basic difficulties (just more complexity). The whole region external to the center conductor and the sheath wires is taken to have the same magnetic permeability μ. In what follows, all field quantities will be taken to vary with time according to $\exp(i\omega t)$.

In accordance with the previous discussion, and for reasons that will become evident below, we adopt the following representation for the current $I(z)$ in the helix at the axial coordinate z:

$$I(z) = \sum_{m=-\infty}^{+\infty} I_m \exp(-i\beta_0 z) \exp\left(-i\frac{2\pi m}{p} z\right) \quad (1)$$

where the summation over m extends over all integers, including zero, from $-\infty$ to $+\infty$. We note here that β_0 is the mean propagation constant in the z direction for the current while p is the axial period or pitch of the helix. The coefficients I_m are to be determined later but for the time being we will consider the fields that result from this helical current.

To facilitate the analysis, we now observe that the components of the surface current density in the cylindrical surface at $\rho = \rho_0$ are

$$j_z(\phi,z) = I(z)\cos\psi(1/\rho_0)\delta(\phi - (2\pi/p)z) \quad (2)$$

and

$$j_\phi(\phi,z) = I(z)\sin\psi(1/\rho_0)\delta(\phi - (2\pi/p)z) \quad (3)$$

where the Dirac or impulse function can be written in its spectral form

$$\delta(\phi - (2\pi/p)z) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} \exp[in(\phi - (2\pi/p)z)] \quad (4)$$

where the summation over n extends over all integers including zero. Thus, on combining (1)–(4), we obtain

$$j_z(\phi,z) = \frac{\cos\psi}{2\pi\rho_0}\sum\sum I_m \exp(-i\beta_{m,n}z)\exp(in\phi) \quad (5)$$

and

$$j_\phi(\phi,z) = \frac{\sin\psi}{2\pi\rho_0}\sum\sum I_m \exp(-i\beta_{m,n}z)\exp(in\phi) \quad (6)$$

where $\beta_{m,n} = \beta_0 + (2\pi/p)(m + n)$. The double infinite summations over m and n, in (5) and (6), and in the subsequent equations are understood.

Field Representations

In general, for a homogeneous region, we can express the vector fields $\mathbf{E}$ and $\mathbf{H}$ in terms of Hertz vectors of the electric type $\mathbf{\Pi}$ and of the magnetic type $\mathbf{\Pi}^*$. Thus

$$\mathbf{E} = -i\mu\omega \operatorname{curl}\mathbf{\Pi}^* + (k^2 + \operatorname{grad}\operatorname{div})\mathbf{\Pi} \quad (7)$$

and

$$\mathbf{H} = i\varepsilon\omega \operatorname{curl}\mathbf{\Pi} + (k^2 + \operatorname{grad}\operatorname{div})\mathbf{\Pi}^* \quad (8)$$

where $k = (\varepsilon\mu)^{1/2}\omega$ is the wavenumber for the homogeneous region under consideration. For cylindrical structures, it usually seems most convenient to choose these so that z components, denoted by Π and Π^*, respectively, are nonvanishing [11]. Then the field components can be obtained from

$$E_\rho = \frac{-i\mu\omega}{\rho}\frac{\partial\Pi^*}{\partial\phi} + \frac{\partial^2}{\partial\rho\,\partial z}\Pi \quad (9a)$$

$$H_\rho = \frac{i\varepsilon\omega}{\rho}\frac{\partial\Pi}{\partial\phi} + \frac{\partial^2}{\partial\rho\,\partial z}\Pi^* \quad (9b)$$

$$E_\phi = i\mu\omega\frac{\partial\Pi^*}{\partial\rho} + \frac{1}{\rho}\frac{\partial^2}{\partial\phi\,\partial z}\Pi \quad (10a)$$

$$H_\phi = -i\varepsilon\omega\frac{\partial\Pi}{\partial\rho} + \frac{1}{\rho}\frac{\partial^2}{\partial\phi\,\partial z}\Pi^* \quad (10b)$$

$$E_z = \left(k^2 + \frac{\partial^2}{\partial z^2}\right)\Pi \quad (11a)$$

$$H_z = \left(k^2 + \frac{\partial^2}{\partial z^2}\right)\Pi^*. \quad (11b)$$

These will be the appropriate forms to employ for the homogeneous region $a < \rho < \rho_0$. In the external region $\rho > \rho_0$, we replace ε by ε_0 and k by k_0 where $k_0 = (\varepsilon_0\mu)^{1/2}\omega$.

Taking a hint from the forms adopted in (5) and (6), we choose

$$\Pi = \sum\sum \Pi_{m,n}\exp(-i\beta_{m,n}z)\exp(in\phi) \quad (12)$$

and

$$\Pi^* = \sum\sum \Pi_{m,n}^*\exp(-i\beta_{m,n}z)\exp(in\phi) \quad (13)$$

where $\Pi_{m,n}$ and $\Pi_{m,n}^*$ are functions of ρ only. Since $(\nabla^2 + k_0^2)\Pi = 0$ in the region $\rho > \rho_0$, it is evident that an appropriate solution is

$$\Pi_{m,n} = A_{m,n}K_n(v_{m,n}\rho) \quad (14)$$

where K_n is the modified Bessel function of the second type of order n and

$$v_{m,n} = (\beta_{m,n}^2 - k_0^2)^{1/2} = i(k_0^2 - \beta_{m,n}^2)^{1/2}.$$

The coefficient $A_{m,n}$ is yet to be determined. In a similar fashion, for $\rho > \rho_0$, we can also write

$$\Pi_{m,n}^* = A_{m,n}^*K_n(v_{m,n}\rho). \quad (15)$$

In seeking the appropriate form of the solutions for the region $a < \rho < \rho_0$, we require that both E_ϕ and E_z should vanish at $\rho = a$. This leads to the adoption of the following forms for this region:

$$\Pi_{m,n} = B_{m,n}Z_n(u_{m,n}\rho) \quad (16)$$

where

$$Z_n(u\rho) = I_n(u\rho) - [I_n(ua)/K_n(ua)]K_n(u\rho)$$

and

$$\Pi_{m,n}^* = B_{m,n}^*Z_n^*(u_{m,n}\rho) \quad (17)$$

where

$$Z_n^*(u\rho) = I_n(u\rho) - [I_n'(ua)/K_n'(ua)]K_n(u\rho).$$

Here

$$u_{m,n} = (\beta_{m,n}{}^2 - k^2)^{1/2} = i(k^2 - \beta_{m,n}{}^2)^{1/2}$$

and we have also introduced the modified Bessel function of the first kind I_n. The prime over the Bessel functions indicates differentiation with respect to the indicated argument or more precisely $I_n'(ua) = dI_n(x)/dx$ evaluated at $x = ua$.

Application of Sheath Boundary Conditions

Now the conditions at the sheath are that the tangential electric fields are continuous and that the tangential magnetic fields are discontinuous by the amount of surface current. An explicit statement is

$$E_z(\rho_0{}^-) = E_z(\rho_0{}^+) \tag{18a}$$

$$H_z(\rho_0{}^-) = H_z(\rho_0{}^+) + j_\phi(\phi,z) \tag{18b}$$

$$E_\phi(\rho_0{}^-) = E_\phi(\rho_0{}^+) \tag{18c}$$

$$H_\phi(\rho_0{}^-) = H_\phi(\rho_0{}^+) - j_z(\phi,z). \tag{18d}$$

Using (9)–(17), these lead easily to the following set of equations:

$$u^2ZB = v^2KA \tag{19a}$$

$$-u^2Z^*B^* + v^2KA^* = J\sin\psi \tag{19b}$$

$$i\mu\omega Z^{*\prime}B^* + (n\beta/\rho_0)ZB = i\mu\omega vK'A^* + (n\beta/\rho_0)KA \tag{19c}$$

$$-i\varepsilon\omega uZ'B + (n\beta/\rho_0)Z^*B^* + i\varepsilon_0\omega vK'A - (n\beta/\rho_0)KA^* = -J\cos\psi \tag{19d}$$

where $A = A_{m,n}$, $B = B_{m,n}$, $A^* = A_{m,n}{}^*$, $B^* = B_{m,n}$, $u = u_{m,n}$, $v = v_{m,n}$, $Z = Z_n(u_{m,n}\rho_0)$, $Z' = Z_n'(u_{m,n}\rho_0)$, $Z^* = Z_n{}^*(u_{m,n}\rho_0)$, $Z^{*\prime} = Z^{*\prime}(u_{m,n}\rho_0)$, $K = K_n(v_{m,n}\rho_0)$, $K' = K_n'(v_{m,n}\rho_0)$, $\beta = \beta_{m,n}$, and $J = I_m/(2\pi\rho_0)$. The four linear equations (19a)–(19d) may be solved explicitly for the coefficients A, B, A^*, and B^* in terms of J. Thus, for example,

$$A = \left[i\mu\omega v\left(\frac{v}{u}\frac{Z^{*\prime}}{Z^*}K - K'\right)\left(\frac{n\beta}{u^2\rho_0}J\sin\psi - J\cos\psi\right) - \frac{n\beta}{\rho_0}K\left(\frac{v^2}{u^2} - 1\right)\frac{i\mu\omega}{u}\frac{Z^{*\prime}}{Z^*}J\sin\psi\right]D^{-1} \tag{20}$$

and

$$A^* = \left[\frac{k_0{}^2v}{u}\frac{Z^{*\prime}}{Z^*}\left(\frac{\varepsilon}{\varepsilon_0}\frac{v}{u}\frac{Z'}{Z}K - K'\right)J\sin\psi - \left(\frac{n\beta}{u^2\rho_0}J\sin\psi - J\cos\psi\right)\frac{n\beta}{\rho_0}K\left(\frac{v^2}{u^2} - 1\right)\right]D^{-1} \tag{21}$$

where

$$D = k_0{}^2v^2\left[\frac{v}{u}\frac{Z^{*\prime}}{Z^*}K - K'\right]\left[\frac{\varepsilon}{\varepsilon_0}\frac{v}{u}\frac{Z'}{Z}K - K'\right] - \left(\frac{n\beta}{\rho_0}K\right)^2\left(\frac{v^2}{u^2} - 1\right)^2. \tag{22}$$

The tangential electric fields in the region external to the sheath (i.e., $\rho > \rho_0$) are given by

$$E_\phi = \sum\sum\left[i\mu\omega v_{m,n}A_{m,n}{}^*K_n'(v_{m,n}\rho) + \frac{1}{\rho}n\beta_{m,n}A_{m,n}K_n(v_{m,n}\rho)\right]\cdot\exp(in\phi)\exp(-i\beta_{m,n}z) \tag{23}$$

and

$$E_z = -\sum\sum v_{m,n}{}^2A_{m,n}K_n(v_{m,n}\rho)\exp(in\phi)\exp(-i\beta_{m,n}z) \tag{24}$$

where the coefficients $A_{m,n}$ and $A_{m,n}{}^*$ are given in terms of the current on the right-handed helix via (21) and (22). Also we should remember that $v_{m,n} = (\beta_{m,n} - k_0{}^2)^{1/2}$ and $\beta_{m,n} = \beta_0 + (2\pi/p)(m + n)$.

To obtain the fields of the current on the corresponding left-handed helix we can proceed precisely in the same fashion. This helix is defined by $\phi = -(2\pi/p)z$ at $\rho = \rho_0$. Also, for the case of usual concern, the current $I(z)$ on this helix will be the same as for the right-handed helix given by (1). The exception discussed later is when the excitation is not locally uniform about the cable. Thus, for this symmetrical situation, the sheath current densities corresponding to (5) and (6) are

$$\hat{\jmath}_z(\phi,z) = \frac{\cos\psi}{2\pi\rho_0}\sum\sum I_m\exp(-i\hat{\beta}_{m,n}z)\exp(in\phi) \tag{25}$$

and

$$\hat{\jmath}_\phi(\phi,z) = -\frac{\sin\psi}{2\pi\rho_0}\sum\sum I_m\exp(-i\hat{\beta}_{m,n}z)\exp(in\phi) \tag{26}$$

where

$$\hat{\beta}_{m,n} = \beta_{m,-n} = \beta_0 + (2\pi/p)(m - n).$$

As indicated, we place a circumflex over the quantity when it refers to the changed form needed for the left-handed helix. The tangential electric fields in the region external to the sheath, that are analogous to (23) and (24), are

$$\hat{E}_\phi = \sum\sum\left[i\mu\omega v_{m,-n}\hat{A}_{m,n}{}^*K_n'(v_{m,-n}\rho) + \frac{1}{\rho}n\beta_{m,-n}\hat{A}_{m,n}K_n(v_{m,-n}\rho)\right]\cdot\exp(in\phi)\exp(-i\beta_{m,-n}z) \tag{27}$$

and

$$\hat{E}_z = -\sum\sum v_{m,-n}^2\hat{A}_{m,n}K_n(v_{m,-n}\rho)\cdot\exp(in\phi)\exp(-i\beta_{m,-n}z). \tag{28}$$

The coefficients $\hat{A}$ and $\hat{A}^*$ are given by (20) and (21) with the reversed sign for ψ (i.e., replace $\sin\psi$ by $-\sin\psi$). We also should note that K and Z are replaced by $\hat{K}$ and $\hat{Z}$ defined by $\hat{K} = K_n(v_{m,-n}\rho_0)$ and $\hat{Z} = Z_n(u_{m,-n}\rho_0)$.

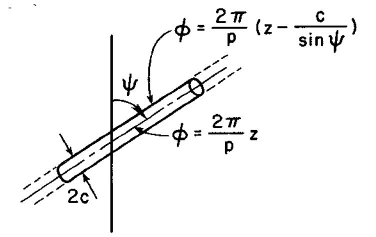

Fig. 2. Microscopic view of segment of helix wire.

It is useful now to note, according to (20) and (21), that

$$A_{m,n} = P_{m,n}I_m \qquad A_{m,n}^* = P_{m,n}^*I_m$$

and

$$\hat{A}_{m,n} = \hat{P}_{m,n}I_m \qquad \hat{A}_{m,n}^* = \hat{P}_{m,n}^*I_m \tag{29}$$

where the P's are explicitly known in terms of the counterwound helix geometry and the specified value of the axial wavenumber β_0.

Application of Wire Boundary Condition

We are now in the position to apply the impedance condition at the helix wires. Since the wires themselves have already been assumed to be very thin, the longitudinal electric field at the surface of the wires is sensibly uniform around the wire circumference. Thus, for convenience, we choose to apply the impedance condition at the top of the wires which by definition is the spiral $z = (p/2\pi)\phi + c/\sin\psi$, where c is the wire radius. This is indicated in the sketch in Fig. 2. Also, because of the assumed rotational symmetry, we need only apply the condition on one helix. The corresponding condition on the other helix will be automatically satisfied. Thus we need to apply

$$[(E_z + \hat{E}_z)\cos\psi + (E_\phi + \hat{E}_\phi)\sin\psi + E_z^p\cos\psi] = I(z)Z_w \tag{30}$$

at $\rho = \rho_0$ and $\phi = (2\pi/p)[z - c/(\sin\psi)]$. Here E_z^p is the axial component of the "primary" field; it is the resultant field that would exist at the surface $\rho = \rho_0$ for the same cylindrical structure but in the absence of the helix wires. As previously mentioned, E_z^p can be regarded as invariant to ϕ when ρ_0 is much less than the free space wavelength (i.e., $k_0\rho_0 \ll 1$). The series impedance per unit length Z_w is determined by the local property of the wires and treated as if they were straight [11]. This appears to be justified always when $c \ll \rho_0$. Thus, if the electrical constants of the wires are σ_w, ε_w, and μ_w, we would use the usual relation

$$Z_w = [\eta_w/(2\pi c)]I_0(\gamma_w c)/I_1(\gamma_w c) \tag{31}$$

where $\eta_w = [i\mu_w\omega/(\sigma_w + i\varepsilon_w\omega)]^{1/2}$ and $\gamma_w = [i\mu_w\omega(\sigma_w + i\varepsilon_w\omega)]^{1/2}$. As $\omega \to 0$ this reduces to the expected dc form, namely $Z_w \to (\sigma_w\pi c^2)^{-1}$.

Using (23), (24), and (27)–(29), the impedance condition (30) now takes the form

$$\begin{aligned}
&\sum_n\sum_m I_mR_{m,n}\exp\left(-in\frac{2\pi c}{p\sin\psi}\right)\exp\left(-i\frac{2\pi}{p}mz\right)\\
&\quad+\sum_n\sum_{m'} I_{m'}\hat{R}_{m',n}\exp\left(-in\frac{2\pi c}{p\sin\psi}\right)\\
&\quad\cdot\exp\left(in\frac{4\pi}{p}z\right)\exp\left(-i\frac{2\pi}{p}m'z\right)\\
&\quad+\sum_m E_z^p\delta_{m,0}\exp\left(-i\frac{2\pi m}{p}z\right)\cos\psi\\
&= Z_w\sum_m I_m\exp\left(-i\frac{2\pi m}{p}z\right)
\end{aligned}\tag{32}$$

where we have used m' in place of m in the second term for convenience in the subsequent manipulation. The coefficients R and $\hat{R}$ are defined by

$$\begin{aligned}
R_{m,n} = &-v_{m,n}^2\cos\psi P_{m,n}K_n(v_{m,n}\rho_0)\\
&+i\mu\omega v_{m,n}\sin\psi P_{m,n}^*K_n'(v_{m,n}\rho_0)\\
&+\frac{\sin\psi}{\rho_0}n\beta_{m,n}P_{m,n}K_n(v_{m,n}\rho_0)
\end{aligned}\tag{33}$$

and

$$\begin{aligned}
\hat{R}_{m,n} = &-v_{m,-n}^2\cos\psi\hat{P}_{m,n}K_n(v_{m,-n}\rho_0)\\
&-i\mu\omega v_{m,-n}\sin\psi\hat{P}_{m,n}^*K_n'(v_{m,-n}\rho_0)\\
&-\frac{\sin\psi}{\rho_0}n\beta_{m,-n}\hat{P}_{m,n}K_m(v_{m,-n}\rho_0).
\end{aligned}\tag{34}$$

Now in (32) we replace m' by $2n + m$, which means that the factor $\exp(-i(2\pi/p)mz)$ is now common to all terms. Then, since (32) is to hold for all z, we obtain

$$\begin{aligned}
&\left[\sum_{n=-\infty}^{+\infty} R_{m,n}\exp\left(-in\frac{2\pi c}{p\sin\psi}\right)\right]I_m\\
&\quad+\left[\sum_{n=-\infty}^{+\infty}\hat{R}_{2n+m,n}\exp\left(-in\frac{2\pi c}{p\sin\psi}\right)\right]I_{2n+m}\\
&\quad+E_z^p\delta_{m0}\cos\psi\\
&= Z_wI_m
\end{aligned}\tag{35}$$

which is to hold for all integer values of m from $-\infty$ to $+\infty$. This, of course, is then an infinite set of equations that must be solved, after appropriate truncation, for the current coefficients I_m.

The Specification of the Primary Field

At this point we should obtain the appropriate expression for the primary field E_z^p that is mentioned previously. The incident plane wave is defined by

$$E = E_0\exp[i\alpha_0\rho\cos\phi]\exp(-i\beta_0 z) \tag{36}$$

where $\alpha_0 = (k_0^2 - \beta_0^2)^{1/2}$. Here we can identify β_0 with $k_0\cos\theta$, where θ is the angle subtended by the wave normal and the negative z axis. Now, in the vicinity of the cable, we can assume for purposes of simplicity that $\alpha_0\rho$ or $k_0\rho \sin\theta$ is much less than one. Also, under the same condition, we only need to be concerned with the axial component of the incident field since the transverse components have a negligible interaction. This leads us to use a quasi-static analysis [4] in order to determine the primary field E_z^p. Thus the required field forms for the coaxial structure, in the absence of the helix wires, are

$$E_z = \alpha^2 \left[P + \frac{2}{\pi} Q \ln 0.89\alpha\rho\right] \quad (37)$$

$$H_\phi = -(2/\pi) i\varepsilon\omega Q/\rho \quad (38)$$

for $a < \rho < \rho_0$

and

$$E_z = E_{0z}[1 + R(1 - i(2/\pi)\ln 0.89\alpha_0\rho)] \quad (39)$$

$$H_\phi = -E_{0z}(2/\pi)\varepsilon_0\omega R/(\alpha_0^2\rho) \quad (40)$$

for $\rho > \rho_0$

where $\alpha = (k^2 - \beta_0^2)^{1/2}$. We now apply the boundary conditions that E_z is zero at $\rho = a$ and that both E_z and H_ϕ are continuous at $\rho = \rho_0$. Thus we readily deduce that

$$P = -(2/\pi) Q \ln 0.89\alpha a \quad (41)$$

$$Q = -i(\varepsilon_0/\varepsilon)(R/\alpha_0^2)E_{0z} \quad (42)$$

and

$$R = -\left[i\frac{2}{\pi}\frac{\varepsilon_0}{\varepsilon}\frac{\alpha^2}{\alpha_0^2}\ln\frac{\rho_0}{a} + 1 - i\frac{2}{\pi}\ln 0.89\alpha_0\rho_0\right]^{-1}. \quad (43)$$

Then, in fact,

$$E_z^p = E_z\Big|_{\rho=\rho_0} = \alpha^2\frac{2}{\pi} Q \ln\frac{\rho_0}{a} \quad (44)$$

or, more explicitly,

$$\frac{E_z^p}{E_{0z}} = \frac{i\dfrac{2}{\pi}\dfrac{\varepsilon_0}{\varepsilon}\dfrac{\alpha^2}{\alpha_0^2}\ln\dfrac{\rho_0}{a}}{1 + i\dfrac{2}{\pi}\left(\dfrac{\varepsilon_0}{\varepsilon}\dfrac{\alpha^2}{\alpha_0^2}\ln\dfrac{\rho_0}{a} - \ln 0.89\alpha_0\rho_0\right)}. \quad (45)$$

Generalizations and Concluding Remarks

There is a generalization, as illustrated in Fig. 3, that we can mention briefly. If we have Q right-handed and Q left-handed spirals, that are equispaced, the formulation is only slightly more involved. For example, the equation for the right-handed helices is

$$\phi = (2\pi/p)(z - (q/Q)) \quad \text{at} \quad \rho = \rho_0 \quad (46)$$

where $q = 0, 1, 2, 3, \cdots, Q - 1$. If the helices are all made of identical wires, then the current on each can still be given by (1) in view of the assumed azimuthal uniformity of the excitation. But now, for example, the z component of the surface current in the sheath for the right-handed

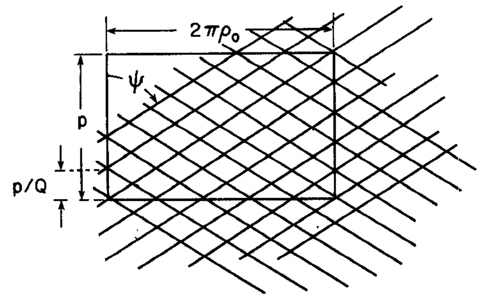

Fig. 3. Planar development of multicounterwound helices (drawn for $Q = 5$).

helices has the form

$$j_z(\phi,z) = I(z)\frac{\cos\psi}{\rho_0}\sum_{q=0}^{Q-1}\delta\left(\phi - \frac{2\pi}{p}z - \frac{2\pi}{Q}q\right) \quad (47)$$

in place of (2). Then, using the spectral representation for the impulse functions, we find that

$$j_z(\phi,z) = \frac{\cos\psi}{2\pi\rho_0}\sum_{q=0}^{Q-1}\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} I_m \exp(-i\beta_{m,n}z) \cdot \exp(-i2\pi nq/Q)\exp(in\phi) \quad (48)$$

with a similar form for $j_\phi(\phi,z)$. Thus, on comparing with (5) and (6), it is evident that the essential modification of the formulation is to introduce the factor $\exp(-i2\pi nq/Q)$ with a summation over the number of separate helices. In dealing with the left-handed helices, we introduce a corresponding factor $\exp(+i2\pi nq/Q)$.

The boundary condition indicated by (30) still may be applied at the one helix only. The resulting coupled equation to determine the coefficients I_m is again given by the following modification of (35):

$$\sum_{q=0}^{Q-1}\left\{\sum_{n=-\infty}^{+\infty} R_{m,n}\exp\left(-in\frac{2\pi c}{p\sin c}\right)\exp\left(-i\frac{2\pi nq}{Q}\right)I_m + \sum_{n=-\infty}^{+\infty} R_{2n+m,n}\exp\left(-in\frac{2\pi c}{p\sin c}\right) \cdot \exp\left(+i\frac{2\pi nq}{Q}\right)I_{2n+m}\right\} + E_z^p\delta_{m,0}\cos\psi = Z_w I_m. \quad (49)$$

Actually, there is some simplification to (48) and (49) by noting that

$$\sum_{q=0}^{Q-1}\exp(\mp i2\pi nq/Q) = \begin{cases} Q, & \text{if } n = lQ \\ 0, & \text{if } n \neq lQ \end{cases} \quad (50)$$

where $l = 0, \pm 1, \pm 2, \pm 3, \cdots$.

Another generalization that does not lead to any basic difficulty, at least for symmetrical excitation, is to remove the assumption of local uniformity of the excitation field. This amounts to representing E_z^p itself as a harmonic expansion in the azimuth direction about the cable axis. Such a modification is only necessary, however, when the cable cross section becomes comparable with a wavelength. In that case, we would also need to account for the presence of the transverse components of the exciting fields, in which case the filamental currents on the counterwound helices are no longer the same. In this situation, we would also need to be concerned with whether the counterwound helices were bonded at their intersections. In analog to the work on planar wire meshes (e.g., Hill and Wait [9]), the difference between bonded and unbonded wire intersections could be significant for the nonsymmetrical component of the excitation field. One method to analyze this situation is to allow the right-handed and the left-handed helices to have slightly different radii. For the symmetrical excitation, which in fact is a good approximation at low frequencies, the final results would not be very sensitive to the difference between the helix radii. Thus we should not expect the bonding to have a major influence on the low-frequency performance of the cable. Nevertheless, this is a subject that should be investigated in a quantitative sense.

The influence of a dielectric jacket and/or lossy external coating on the cable can be considered in a straightforward manner. Basically, this amounts to introducing radial wave functions in the region external to the sheath that satisfy the appropriate boundary conditions at the one or more new cylindrical interfaces. Finally, we should mention that the corresponding natural modes of propagation on the composite structure are obtained by simply letting the incident field be zero and then setting the (infinite) determinant of the coefficients of I_m in (35) or (49) to zero and solving for the propagation constant(s) $i\beta_0$. Such solutions would include the surface waves that have their energy confined to the region of the sheath.

In Part II, we consider the numerical aspects of this general problem and the results are applied to specific cable configurations.

Appendices

A. A Note on Convergence

The series over n have the following form:

$$S = \sum_{n=-\infty}^{+\infty} A_n \exp\left[-in\frac{2\pi c}{p\sin\psi}\right]. \quad (51)$$

The convergence may be very poor since c is small. Thus, as in other similar problems, there is some merit in summing the higher order terms in closed form by making use of the fact that the coefficient A_n admits to an asymptotic expansion of the type

$$\lim_{(|n|\to\infty)} A_n \sim \pm n\left(A^{(0)} + \frac{A^{(1)}}{n^2} + \cdots\right). \quad (52)$$

This suggests that we write

$$S = A_0 + \sum_{n=1}^{\infty}\left(A_n - nA^{(0)} - \frac{A^{(1)}}{n}\right)\exp\left[-in\frac{2\pi c}{p\sin\psi}\right] + \sum_{n=1}^{\infty}\left(A_{-n} - nA^{(0)} - \frac{A^{(1)}}{n}\right) \cdot \exp\left[+in\frac{2\pi c}{p\sin\psi}\right] + \Delta S \quad (53)$$

where

$$\Delta S = 2A^{(1)}\sum_{1}^{\infty}\frac{1}{n}\cos n\frac{2\pi c}{p\sin\psi} = -2A^{(1)}\ln\left(2\sin\frac{2\pi c}{p\sin\psi}\right). \quad (54)$$

Here we have utilized the fact that

$$\sum_{n=-\infty}^{+\infty} n\exp(\pm inx) = 0$$

for all real $x > 0$.

B. Field Averaging

In general, a field component ψ has the following doubly infinite series representation

$$\psi = \sum_{n=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}\psi_{m,n}\exp(-i\beta_{m,n}z)\exp(in\phi) \quad (55)$$

where $\psi_{m,n}$ is a coefficient that does not depend on the coordinates ϕ and z. Now when dealing with a coaxial cable with electrically small radius and for the case where the axial period of the braid is small, the far field scattered from the cable only involves the term for $n = m = 0$. It also follows rather simply that the "average" field ψ near or at the cable also can be described by this term. This follows from the fact that

$$\bar{\psi} = \frac{1}{p}\int_0^p\left[\frac{1}{2\pi}\int_0^{2\pi}\psi\,d\phi\right]\exp(i\beta_0 z)\,dz = \psi_{0,0}. \quad (56)$$

In view of this reasoning, it follows that a suitable definition of the *effective* axial impedance $Z_e(i\beta_0)$ of the cable is

$$Z_e(i\beta_0) = \bar{E}_z/(2\pi\rho\bar{H}_\phi)\Big|_{\rho=\rho_0} \simeq \frac{E_{0z} - v_{0,0}{}^2A_{0,0}K_0(v_{0,0}\rho_0)}{2\pi i\varepsilon_0\omega A_{0,0}} \quad (57)$$

where $v_{0,0} = i\alpha_0$. This quantity is a useful description of the cable when its behavior in a more complicated environment is to be considered. It is stressed that $Z_e(i\beta_0)$ is a function of axial wavenumber.

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Vector Variational Formulation of Electromagnetic Fields in Anisotropic Media

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Abstract—**Maxwell's equations can be cast into a basic differential operator equation, the curlcurl equation, which lends itself easily to variational treatment. Various forms of this equation are associated with problems of practical importance. The formulation includes the treatment of loss-free anisotropic media. The boundary conditions associated with electromagnetic-field problems are treated in detail and the uniqueness of the solution is discussed. A functional is derived for the curlcurl equation in Cartesian and cylindrical coordinates.**

I. Introduction

DUE TO the broad variety of practical applications of waveguides, resonators, and other microwave devices, the development of methods to solve the associated electromagnetic-field problems has received a great deal of attention in the past two decades. Such electromagnetic boundary value problems, with the exception of isotropic waveguides, require a formulation in which the electric and magnetic fields are treated as vector quantities. In recent years, a variety of methods for the solution of homogeneous isotropic waveguide problems appeared in the literature; these have been reviewed by Wexler [1], by Davies [2], and by Ng [3]. With a few exceptions [4]–[9], [29], the tendency in recent years was to formulate the inhomogeneous isotropic waveguide problem in terms of the longitudinal electric (E_z) and magnetic (H_z) field components [10]–[18].

As noted by Wexler [1] in 1969, there have been many proponents and only a few attempts to formulate electromagnetic-field problems in terms of all three components of the field vectors. Among the attempts one must mention Harrington's well-known monograph [4] and Gupta's doctoral dissertation [5] on field solution in resonant cavities filled with an inhomogeneous anisotropic medium. The moment method employed by these authors is essentially a projective method in which the field components in a cavity or waveguide are expanded in terms of the field components of the empty cavity or waveguide modes.

In 1967 Hannaford [10] proposed an extension of his variational/finite difference method for homogeneous isotropic waveguides to plasma- and ferrite-filled waveguides. Hannaford's proposal involves only the longitudinal field components. For inhomogeneous media, the resulting coefficient matrix in Hannaford's formulation becomes indefinite above the 45° "air-line" on the dispersion diagram. Since 1967, this shortcoming of two-component formulations has reoccurred in a number of other finite-difference and finite-element variational methods [11]–[17]. Hannaford dismissed Berk's often quoted variational expressions which were published in 1956 [6] as being more complicated than the E_z–H_z formulation. Berk derived three- and six-component vector variational expressions in the form of Rayleigh quotients for the resonance frequencies of a resonator filled with loss-free, anisotropic, homogeneous or inhomogeneous media.

The only three-component vector variational formulation for electromagnetic-field problems appearing in recent years is due to English and Young [7]. They select the $\boldsymbol{E}$-field formulation over $\boldsymbol{H}$ on the basis of the number of Dirichlet boundary conditions to be satisfied. The authors list the advantages of the three-component vector formulation as reduced matrix size and denser coefficient matrices in comparison with the six-component formulation given by English in his doctoral dissertation [8] and in two papers by English [9], [23] which appeared in 1971. However,

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